Collision Theory for Plasmas

Recall Boltzmann Equation:

$$\frac{df(p)}{dt} = \frac{\partial f}{\partial t} + \frac{p}{m}\cdot\frac{\partial f}{\partial x} + q\cdot\frac{\partial f}{\partial p} = C(f)$$

1 2

$$C(f) = \int dp_1 \int dp' \int dp_1' \underbrace{W}_{\text{transition probability}} \left(f(p')\, f(p_1') - f(p)\, f(p_1) \right)$$

Now $\int C(f)\, dp = 0$ $\left\{ \begin{array}{l} \text{one integration trivial} \\ p + p_1 = p' + p_1' \end{array} \right.$

so can write: $C(f) = -\frac{\partial \cdot \underline{J}(p)}{\partial p}$ → "current" in momentum or velocity...

$$\int C(f) = \int -\frac{\partial J}{\partial p} = 0$$

Now, for slight deflection:

- can proceed by small momentum transfer limit of B. E. i.e. $p \to p + k/2$, $p' \to p' - k/2$

or

- directly from F.P.E., ala' HW. 1.

Now, consider terms in B.E. :

i.e. term ②, with $p_1 \to p'$ (re-label)

$$W(p + q/2,\ p' - q/2,\ q) \langle f(p) \rangle \langle f(p') \rangle d^3p' d^3q$$

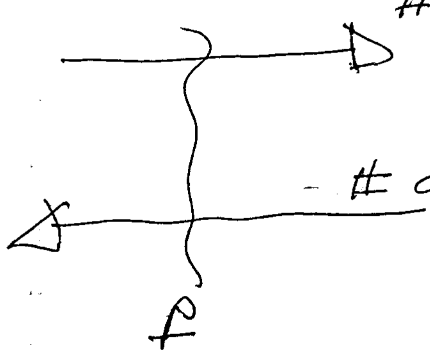
\# of collisions per unit time between particle with momentum p and particle with momentum p' in d^3p'

i.e. $p \to$ test particle $\to$ scatter-ee

$p' \to$ bckgnd or field particle $\to$ scatter-er

$q \equiv$ momentum transfer

Now, for $\Gamma(p)$ - sum RL + LR



\# left-rt. i.e. increase in momentum

\- # decreasing momentum rt-left

ee, ei' etc

$$\# LR = \sum_{\text{spec}}^{\delta} \int_{|q|\geq 0} d^3q \int d^3p' \int_{p-q}^{p} w(p+\tfrac{q}{2}, p'-\tfrac{q}{2}, q) \, *$$

$$\langle f(p')\rangle \langle f(p)\rangle$$

$$\# RL = \sum_{\text{spec}} \int_{|q|>0} d^3q \int d^3p' \int_{p-q}^{p} w(p+\tfrac{q}{2}, p'-\tfrac{q}{2}, -q) \, *$$

$$\langle f(p+q)\rangle \langle f'(p'-q)\rangle$$

Note:

- detailed balance $\Leftrightarrow$ symmetry w/r final $\leftrightarrow$ initial interchange

$$\Rightarrow \; w(p+q/2, p'-q/2, q) = w(p+q/2, p'-q/2, -q)$$

- small deflection:

$$w(p+q/2, p'-q/2, q) \approx w(p, p', q)$$

Then, for q small:

$$\nu(\underline{p}) = \sum_{\text{spec.}} \int_{|q| \geq 0} d^3 q \int d^3 p' \int_{p'-q}^{p} w \, *$$

$$\left[\langle f(\underline{p}') \rangle \langle f(\underline{p}) \rangle - \langle f(\underline{p}+\underline{q}) \rangle \langle f(\underline{p}'-\underline{q}) \rangle \right]$$

$$- \langle f(\underline{p}') \rangle \langle f(\underline{p}) \rangle$$

$$\cong (\ \ ) \left[\cancel{\langle f(\underline{p}') \rangle \langle f(\underline{p}) \rangle} - \underline{q} \cdot \frac{\partial \langle f(\underline{p}) \rangle}{\partial \underline{p}} \langle f(\underline{p}') \rangle + \underline{q} \cdot \frac{\partial \langle f(\underline{p}') \rangle}{\partial \underline{p}} \langle f(\underline{p}) \rangle \right]$$

$$= (\ \ ) \left[\langle f(\underline{p}) \rangle \frac{\partial \langle f(\underline{p}') \rangle}{\partial p'_\beta} - \langle f(\underline{p}') \rangle \frac{\partial \langle f(p) \rangle}{\partial p_\beta} \right] q_\beta$$

Now $(\ \ ) = [\ ] \int_{p_\alpha - q_\alpha}^{p_\alpha} = q_\alpha$

and writing $w \, d^3 q = |\underline{v} - \underline{v}'| \, d\sigma$

$\Rightarrow$

$$\underline{J}(\underline{p}) = \sum \int d^3 p' \left[\langle f(\underline{p}) \rangle \frac{\partial}{\partial p'_\beta} \langle f(\underline{p}') \rangle - \langle f(\underline{p}') \rangle \frac{\partial \langle f(\underline{p}) \rangle}{\partial p_\beta} \right] B_{\alpha\beta}$$

$$B_{\alpha\beta} = \int \frac{d^3 q}{2} |\underline{v} - \underline{v}'| \, q_\alpha q_\beta \, d\sigma$$

↳ prevent dbl counting

↳ momentum current depends on gradients only .....

Note that then can write:

$$\frac{df}{dt} = -\frac{\partial}{\partial \underline{p}} \cdot \underline{J}(\underline{p})$$

$$= \frac{\partial}{\partial \underline{p}} \cdot \left[\underline{\underline{D}} \cdot \frac{\partial}{\partial \underline{p}} \langle f(\underline{p}) \rangle - \underline{F} \langle f(\underline{p}) \rangle \right]$$

$$\underline{\underline{D}} = \sum_{spc} \int d^3p' \langle f(\underline{p}') \rangle \underline{\underline{B}}_{\alpha\beta}$$ → diffusive scattering by field particles

i.e. test particle diffused by background particles

$$\underline{F} = \sum_{spc} \int d^3p' \frac{\partial \langle f(\underline{p}') \rangle}{\partial p'_\beta} \underline{\underline{B}}_{\alpha\beta}$$ → ?? → drag,

i.e. dynamical friction exerted on test particle by field particles.

Obviously, from Fokker-Planck approach:

$$\frac{df}{dt} = -\frac{\partial}{\partial \underline{p}} \cdot \left[\frac{\langle \Delta \underline{p} \rangle}{\Delta t} f - \frac{\langle \Delta \underline{p} \Delta \underline{p} \rangle}{2 \Delta t} \cdot \frac{\partial f}{\partial \underline{p}} \right]$$

Clearly:

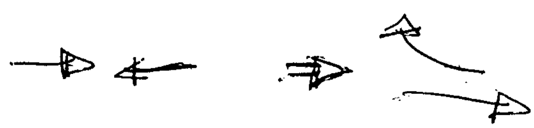
$$\underline{D} \leftrightarrow \frac{\langle \Delta p \, \Delta p \rangle}{2 \Delta t}$$

$$\underline{F} \leftrightarrow \left\langle \frac{\Delta p}{\Delta t} \right\rangle = \left\langle \frac{dp}{dt} \right\rangle$$

To simplify:

- $|\underline{q}|$ small $\leftrightarrow$ small $\measuredangle$ scattering

$$\underline{q} \perp \underline{v} - \underline{v}'$$



so

- $$B_{\alpha\beta}(v_\beta - v'_\beta) = 0$$

so since $\underline{\underline{B}}$ transverse $\underline{v}-\underline{v}'$ and depends on $\underline{v}-\underline{v}'$;

$$B_{\alpha\beta} = \frac{B}{2}\left[\delta_{\alpha\beta} - \frac{(v_\alpha - v'_\alpha)(v_\beta - v'_\beta)}{(v-v')^2}\right]$$

$$B = \frac{1}{2}\int q^2 \, |\underline{v} - \underline{v}'| \, d\sigma = B_{\alpha\alpha}$$

For tensor $B_{\alpha\beta}$:

χ = angle deviation $\underline{V}-\underline{V}'$ (upon collision)
$\downarrow$ deflection

$q \cong \mu\,|\underline{V}-\underline{V}'|\,\chi$
$\downarrow$
reduced mass

$\underline{\underline{so}}$

$$B \equiv \frac{1}{2}\int q^2\,|\underline{V}-\underline{V}'|\,d\sigma$$
$$\cong \frac{1}{2}\int d\sigma\left(\mu^2|\underline{V}-\underline{V}'|^2\chi^2\right)|\underline{V}-\underline{V}'|\,d\sigma$$

$$\Rightarrow \quad B = \frac{\mu^2}{2}|\underline{V}-\underline{V}'|^3\int \chi^2\,d\sigma$$

$$= \mu^2|\underline{V}-\underline{V}'|^3\sigma_T$$

$\sigma_T = \int(1-\cos\chi)\,d\sigma$ $\rightarrow$ transport cross-section

$$\cong \int(\chi^2/2)\,d\sigma$$

Now, for Rutherford / Coulomb scattering:

$$d\sigma = \frac{4(qq')^2\, d\Omega}{\mu^2 |\underline{v}-\underline{v}'|^4 \chi^4}$$

$$\cong \frac{8\pi (qq')^2}{\mu^2 |\underline{v}-\underline{v}'|^4} \frac{d\chi}{\chi^3}$$

$$\sigma_T = \int \frac{\chi^2}{2}\, d\sigma$$

$$= \int \frac{\chi^2}{2}\, 8\pi \frac{(qq')^2}{\mu^2 |\underline{v}-\underline{v}'|^4} \frac{d\chi}{\chi^3}$$

$$= \frac{4\pi (qq')^2}{\mu^2 |\underline{v}-\underline{v}'|^4} \underbrace{L}_{\text{Coulomb Logarithm } (\Lambda)}$$

$$L = \int \frac{d\chi}{\chi} = \ln\left(\chi_{max}/\chi_{min}\right)$$

$$\cong \ln\left(1/\chi_{min}\right)$$

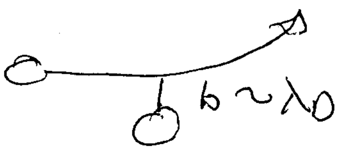
→ divergence! → origin is <u>small-angle</u> collisions.

- known as "Coulomb logarithm".

Now to estimate Coulomb logarithm:

→ particle interaction screened at distance λ_D by shielding cloud (collective response enters!) → Debye length sets screening scale.

χ_{min} → scattering angle for passage at impact parameter λ_D



so $\quad |q| \sim \underset{e^2}{(\text{Force})}\,(\text{interaction time})_{e^2}$

$$\sim \frac{|qq'|}{\lambda_D^2}\;\frac{\lambda_D}{|V_{rel}|} \sim \frac{|qq'|}{\lambda_D\,|V_{rel}|}$$

$$|q| \sim (\text{Momentum})(\chi)$$

$$\sim qq' / \lambda_D\,|V_{rel}|$$

so

$$\frac{e^2}{\lambda_D\,|V_{rel}|} \sim (\mu\, V_{rel})\,\chi_{min}$$

$\Rightarrow$

$$\Rightarrow \quad x_{min} \simeq |ee'| / \lambda_D \mu v_{rel}^2$$

$$\therefore \quad L = \ln \Lambda = \ln \left(\lambda_D \mu v_{rel}^2 / |ee'| \right)$$

Alternative $\rightarrow$ Test Particle Model

$$\sigma = \frac{4\pi (ee')^2}{\mu^2 (\underline{v}-\underline{v}')^4} L \; , \quad L = \int dx/x$$

$L \to$ Coulomb Logarithm $\to$ cut-off

$$B_{\alpha\beta} = \frac{2\pi (ee')^2}{|\underline{v}-\underline{v}'|} L \left[\delta_{\alpha\beta} - \frac{(v_\alpha - v_\alpha')(v_\beta - v_\beta')}{(\underline{v}-\underline{v}')^2} \right]$$

Key elements:

→ glancing collisions / weak deflections

→ molecular chaos, i.e. $f(1,2) = f(1)\, f(2)$

So, natural to compare:

	Landau Collision Theory	Test Particle Model; Lenard-Balescu Theory
"Scenario"	(test) particle scattered by equilibrium distribution of field particles.	test particle scattered by screened ballistic spectrum $v \sim \omega/k$, due <u>other</u> particles.
Correlation	uncorrelated test, field particles via molecular chaos $\langle f(1,2)\rangle = \langle f(1)\rangle\langle f(2)\rangle$	discrete uncorrelated test particles $\langle \tilde{f}\tilde{f}\rangle = \frac{\langle f\rangle}{n}\delta(x_-)\,\delta(v_-)$
Collective effects	no screening $\Rightarrow$ must cut-off Coulomb logarithm "by hand"	screening via f^c and $1/\lvert\epsilon(k,\omega)\rvert^2 \Rightarrow$ screening length appears naturally
nonlinearity	weak deflection $\lvert g\rvert \ll \lvert f\rvert$	unperturbed orbits

suggests that incorporation of screening into Landau collision integral should recover L-B. theory.

→ Recall Landau Collision integral:

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial p_\alpha} J_\alpha$$

detailed balance small momentum transfer $|q|$ ⇒

$$J_\alpha = \sum_{spc.} \int B_{\alpha\beta} \Big[\underbrace{f(p) \frac{\partial f(p')}{\partial p'_\beta}}_{\text{drag}} - \underbrace{f(p') \frac{\partial f(p)}{\partial p_\beta}}_{\text{diffusion}} \Big] d^3p'$$

$p' \equiv$ field particle
$p \equiv$ test particle
> momenta

$$B_{\alpha,\beta} = \frac{1}{2} \int d\sigma \, q_\alpha q_\beta |\underline{v} - \underline{v}'|$$

$$= \frac{2\pi (ee')^2}{|\underline{v} - \underline{v}'|} L \Big[\delta_{\alpha\beta} - \underbrace{\frac{(v_\alpha - v_\alpha')(v_\beta - v_\beta')}{(v - v')^2}}_{\text{transverse form.}} \Big]$$

$\underline{q \perp v}$

$d\sigma = 2\pi b \, db$

$$\frac{ee'}{b} \sim \frac{\mu v_{rel}^2}{2} \Rightarrow b^2 \sim 4 \left[\frac{(ee')}{\mu v_{rel}^2} \right]^2$$

Recall; $C(f) = -\frac{\partial}{\partial \underline{p}} \cdot \underline{\Gamma}(\underline{p})$

$$\underline{\Gamma}_\alpha(\underline{p}) = \sum_{s,c} \int d^3p' \left[\frac{\partial}{\partial p'_\beta} \langle f(p') \rangle \langle f(p) \rangle - \langle f(p') \rangle \frac{\partial}{\partial p_\beta} \langle f(p) \rangle \right] B_{\alpha\beta}$$

$\sigma \sim \pi b^2$

$l_{mfp} \sim 1/n\sigma$

$\nu = V_{Th}/l_{mfp}$

$$d\sigma = \frac{4(ee')^2 d\rho}{\mu^2 \, V_{rel}^4 \, \chi^4}$$

$$= \frac{4(ee')^2 d\chi}{\mu^2 V_{rel}^4 \chi^3}$$

$$\therefore \quad d\sigma_{trans} = \chi^2 d\sigma$$

To connect Landau Collision theory and Lenard-Balescu Theory, calculate $B_{\alpha\beta}$ in test particle model and recover screened Landau result.

So

→ calculate $\hat{\phi}$ due to screened test particle of velocity $\underline{v}'$

→ calculate deflection of particle of velocity $\underline{v}$ due $\hat{\phi}$.

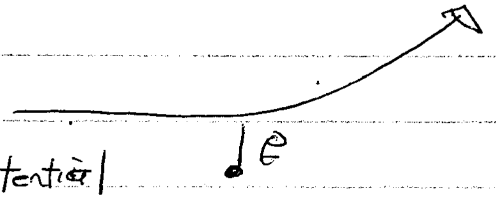
Now $-\nabla \cdot \underline{\underline{\epsilon}} \, \nabla \phi = 4\pi e' \, \delta(\underline{x} - \underline{v}'t)$

$$\hat{\phi}_{k,\omega} = \frac{4\pi e'}{k^2 \epsilon(k,\omega)} 2\pi \, \delta(\omega - \underline{k}\cdot\underline{v}')$$

$$\phi_k(t) = \int \frac{d\omega}{(2\pi)} \frac{4\pi e'}{k^2 \epsilon(k,\omega)} 2\pi\, \delta(\omega - \underline{k}\cdot\underline{v}')\, e^{-i\omega t}$$

$$= \frac{4\pi e'}{k^2 \epsilon(k, \underline{k}\cdot\underline{v}')} e^{-i\underline{k}\cdot\underline{v}' t}$$

For deflection :



interaction potential

$$\underline{q} = \int_{u.p.o.} dt \left(- \frac{\partial U}{\partial \underline{r}} \right) = - \int_{\underline{r} = \rho + \underline{v} t} \frac{\partial U}{\partial r}$$

↳ impact param.

$$U = e\phi$$

$$= 4\pi e e' \int_{upo} \frac{d^3k\, e^{i\underline{k}\cdot\underline{r}}\, e^{-i k\cdot \underline{v}' t}}{k^2 \epsilon(\underline{k}, \underline{k}\cdot\underline{v}')}$$

$$= 4\pi e e' \int \frac{d^3k\, e^{i\underline{k}\cdot\rho}\, e^{i\underline{k}\cdot(\underline{v}-\underline{v}')t}}{k^2 [\epsilon(\underline{k}, \underline{k}\cdot\underline{v}')]}$$

$$\underline{q} = 4\pi e e' \int \frac{d^3k}{(2\pi)^3} \frac{-i\underline{k}\, e^{i\underline{k}\cdot\rho}}{k^2 \epsilon(\underline{k}, \underline{k}\cdot\underline{v}')} 2\pi\, \delta(\underline{k}\cdot(\underline{v}-\underline{v}'))$$

from:

$$\int dt\, e^{i\underline{k}\cdot(\underline{v}-\underline{v}')t}$$

using $\delta(\underline{k}\cdot(\underline{v}-\underline{v}')) = \delta(k_{\parallel}\,|\underline{v}-\underline{v}'|)$

$$= \frac{1}{|\underline{v}-\underline{v}'|}\,\delta(k_{\parallel})$$

$\therefore$

$$\boxed{\underline{q} = 4\pi e e' \int \frac{d^2 k_\perp}{(2\pi)^2}\,\frac{-i\underline{k}_\perp\, e^{i\underline{k}_\perp\cdot\underline{\rho}}}{k^2\,\epsilon(\underline{k},\underline{k}\cdot\underline{v}')\,|\underline{v}-\underline{v}'|}}$$

momentum transfer

for $B_{\alpha,\beta}$

$$B_{\alpha\beta} = \int d\nabla_{+}\,\frac{q_\alpha q_\beta}{2}\,|\underline{v}-\underline{v}'|$$

$$= \int d^2\rho\; q_\alpha q_\beta\,|\underline{v}-\underline{v}'|$$

$$\int d^2\rho\, q_\alpha q_\beta \sim \int d^2\rho\; e^{i\underline{k}_\perp\cdot\underline{\rho}}\, e^{i\underline{k}_\perp'\cdot\underline{\rho}}$$

$$\sim (2\pi)^2\,\delta(\underline{k}_\perp + \underline{k}_\perp')$$

$$\int d^2 k_\perp'\,\delta(\underline{k}_\perp + \underline{k}_\perp') = 1$$

so

$$B_{\alpha,\beta} = 2e^2e'^2 \int d^2k_\perp \frac{\underline{k}_{\perp\alpha}\,\underline{k}_{\perp\beta}}{|k_\perp^2\,\epsilon(\underline{k}_\perp, \underline{k}_\perp\cdot\underline{v}')|^2\,|\underline{v}-\underline{v}'|}$$

$\Rightarrow$

$$\boxed{B_{\alpha,\beta} = 2e^2e'^2 \int d^2k_\perp \frac{k_{\perp\alpha}\,k_{\perp\beta}}{|k_\perp^2\,\epsilon(\underline{k}_\perp, \underline{k}_\perp\cdot\underline{v}')|^2\,|\underline{v}-\underline{v}'|}}$$

Note:

i.) $\epsilon(\underline{k}, \underline{k}\cdot\underline{v}')$ $\to$ dynamic screening factor
$\to$ evaluates ϵ induced by (ballistically) propagating source

ii.) note if $\epsilon \to 1$ (no collective screening)

$$B \sim \int d^2k \frac{k_\perp^2}{|\epsilon|^2 k_\perp^4} \sim \int dk_\perp\, k_\perp \frac{k_\perp^2}{k_\perp^4} \sim \int \frac{dk_\perp}{k_\perp}$$

$$\sim \ln(k_{\perp max}/k_{\perp min})$$

$\to$ recovers Coulomb logarithm.

if $\underline{k}, \omega \to 0$

$$\epsilon = 1 + 1/k^2\lambda_D^2$$

$k_\perp^2\epsilon \sim k_\perp^2 + 1/\lambda_D^2$ $\to$ $\underline{\underline{\text{no}}}$ long range divergence iii.) limits of integration:

$k_{min} \sim 1/\lambda_D$ (via ϵ)

$k_{max} \sim \frac{1}{2} \mu v_{rel}^2 / ee'$ (distance of closest approach)

Now, can re-write $B_{\alpha,\beta}$ as

$$B_{\alpha,\beta} = 2(ee')^2 \int_{-\infty}^{+\infty} d\omega \int_{k<k_{max}} d^3k \; \delta(\omega - \underline{k}\cdot\underline{v})\,\delta(\omega - \underline{k}\cdot\underline{v}') \; \frac{k_\alpha k_\beta}{k^4 |\epsilon(k,\omega)|^2}$$

recovers L-B, theory noting:

one $\delta(\omega - \underline{k}\cdot\underline{v}')$ $\Rightarrow$ propagator in $\langle \tilde{f}\tilde{f} \rangle_{k,\omega}$

2nd $\delta(\omega - \underline{k}\cdot\underline{v})$ $\Rightarrow$ propagator in Q.L. term.

→ Properties of Landau Collision Integral

n.b.: Read Kulsrud 8.1 → 8.3 ⟶ different approach

now, switching from $p \to \underline{v}$

$$\left.\frac{\partial \langle f \rangle}{\partial t}\right|_c = -\frac{\partial}{\partial \underline{v}} \cdot \underline{J}(\underline{v})$$

$$\underline{J}(\underline{v}) = \sum_{spc} \int d^3v' \, B_{\alpha\beta} \left[\frac{\partial f(\underline{v}')}{\partial v'_\beta} f(\underline{v}) - f(\underline{v}') \frac{\partial f(\underline{v})}{\partial v_\beta} \right]$$

$$B_{\alpha\beta} = \frac{2\pi (q q')^2}{\mu_p^2 |v - v'|} \ln \Lambda \left[\delta_{\alpha\beta} - \frac{v_{\alpha rel} v_{\beta rel}}{v_{rel}^2} \right]$$

$$v - v' = v_{rel}$$

then for electrons,

$$\left.\frac{\partial \langle f \rangle}{\partial t}\right|_c = -\frac{\partial}{\partial v} J(\underline{v})$$

$$= -\frac{\partial}{\partial \underline{v}} \cdot \left[\underbrace{J_{e,e}(\underline{v})}_{\text{electrons as field particles}} + \underbrace{J_{e,i}(\underline{v})}_{\text{ions as field particles}} \right]$$

$$J(v)_\alpha = \int d^3v'_e \, B_{\alpha\beta}^{e,e} \left[\frac{\partial f_e(\underline{v}'_e)}{\partial v'_{e\beta}} f_e(v) - f_e(v'_e) \frac{\partial f_e(\underline{v})}{\partial v_\beta} \right]$$

$$+ \int d^3v'_i \, B_{\alpha\beta}^{e,i} \left[\frac{\partial f_i(v'_i)}{\partial v'_{i\beta}} f_e(\underline{v}) - f_i(v'_i) \frac{\partial f_e(\underline{v})}{\partial v_\beta} \right]$$

$$B_{\alpha\beta}^{e,e} = \frac{2\pi e^2}{m_{e,e} |v_{rel}|} \ln \Lambda_{e,e} \left[\delta_{\alpha\beta} - \frac{v_{rel\,\alpha} v_{rel\,\beta}}{v_{rel}^2} \right]$$

$$B_{\alpha\beta}^{e,i} = \frac{2\pi e^2}{m_{e,i} |v_{rel}|} \ln \Lambda_{e,i} \left[\delta_{\alpha\beta} - \frac{v_{rel\,\alpha} v_{rel\,\beta}}{v_{rel}^2} \right]$$

⇝ negligible difference in magnitude.

Note can simplify form to:

$$\left.\frac{\partial f}{\partial t}\right)_c = \underbrace{\frac{2\pi e^4 \ln \Lambda}{m^2}}_{\text{lead factor}} \frac{\partial}{\partial \underline{v}} \cdot \underbrace{\underline{J}(\underline{v})}_{\text{current (sign flipped)}}$$

$$\underline{J}(\underline{v}) = \int d^3v' \left(\frac{\underline{\underline{I}} - \underline{g}\,\underline{g}}{g} \right) \cdot \left[\underbrace{\frac{\partial f(\underline{v})}{\partial \underline{v}} f(\underline{v}')}_{\text{diffn}} - \underbrace{\frac{\partial f(\underline{v}')}{\partial \underline{v}'} f(\underline{v})}_{\text{drag}} \right]$$

Now, need demonstrate several properties:

(a)
- H theorem

(b)
- conservation for like species (ala' L-B.)

Now, H theorem:

→ entropy increases, except for Maxwellian

→ $\frac{dH}{dt} \leq 0$ $H = \int f \ln f \, d^3v$
$= -S$

need show
H ↓ for S ↑

$$\frac{dH}{dt} = \int d^3v \left[C(f) \ln f + C(f) \right]$$

$$C(f) = + \underline{\nabla}_v \cdot \underline{J}$$

$$\frac{dH}{dt} = \int d^3v \left[C(f) \ln f + C(f) \right]$$

$$= \int d^3v \left[\nabla_v \cdot \underline{J}(\underline{v}) \ln f + \cancel{\nabla_v \cdot \underline{J}(\underline{v})} \right]$$

\# conservation

$$= - \int d^3v \, \frac{1}{f} \frac{\partial f}{\partial \underline{v}} \cdot \underline{J}(\underline{v})$$

so

$$\frac{dH}{dt} = (\#) - \int d^3v \int d^3v' \, \frac{1}{f} \frac{\partial f}{\partial \underline{v}} \cdot \left(\frac{\underline{\underline{I}} - \underline{g}\,\underline{g}}{g} \right) *$$

$$\left[\frac{\partial f(\underline{v})}{\partial \underline{v}} f(\underline{v}') - \frac{\partial f(\underline{v}')}{\partial \underline{v}'} f(\underline{v}) \right]$$

now $\underline{v} \Leftrightarrow \underline{v}'$ and add $\Rightarrow$ $\underline{J}$ is odd in interchange

so

$$\frac{dH}{dt} = (\#) \int d^3v \int d^3v' \left(\frac{1}{f(\underline{v})} \frac{\partial f}{\partial \underline{v}} - \frac{1}{f(\underline{v}')} \frac{\partial f}{\partial \underline{v}'} \right) \cdot \left(\frac{\underline{\underline{I}} - \underline{g}\,\underline{g}}{g} \right) \cdot$$

$$\left[\frac{\partial f(\underline{v})}{\partial \underline{v}} f(\underline{v}') - \frac{\partial f(\underline{v}')}{\partial \underline{v}'} f(\underline{v}) \right] (-1)$$

$$= \# \int d^3v \int d^3v' \, \frac{1}{f(\underline{v}) f(\underline{v}')} \left(f(\underline{v}') \frac{\partial f}{\partial \underline{v}} - f(\underline{v}) \frac{\partial f}{\partial \underline{v}'} \right) \cdot$$

$$\left(\frac{\underline{\underline{I}} - \underline{g}\,\underline{g}}{g} \right) \cdot \left[\frac{\partial f(\underline{v})}{\partial \underline{v}} f(\underline{v}') - \frac{\partial f(\underline{v}')}{\partial \underline{v}'} f(\underline{v}) \right] (-1)$$

$$= - \int d^3v' \int d^3v \, \frac{1}{f(\underline{v}) f(\underline{v}')} \left(\quad \right) \cdot \left(\frac{\underline{\underline{I}} - \underline{g}\,\underline{g}}{g} \right) \cdot \left(\quad \right)$$

where $\left(\;\right] = \left[f(\underline{v}')\dfrac{\partial f}{\partial \underline{v}} - f(\underline{v})\dfrac{\partial f}{\partial \underline{v}'} \right]$

so $\dfrac{dH}{dt} < 0 \Rightarrow$ entropy increases.

Note $\rightarrow \dfrac{dH}{dt} = 0$ for Maxwellian.

$\rightarrow$ if e, i interaction, note $\dfrac{dH}{dt} = 0$ if both electrons, ions are Maxwellian.

(b) For energetics,

$$E = \int \frac{1}{2} m v^2 f \, d^3v$$

$$\left.\frac{d}{dt} E\right|_{e-e} = \frac{2\pi e^4}{m^2} \ln\Lambda \int d^3v \frac{m v^2}{2} \frac{\partial}{\partial \underline{v}} \cdot \int \frac{d^3v'}{g} \left(\underline{\underline{I}} - \hat{g}\hat{g}\right) \left(\frac{\partial f}{\partial \underline{v}} f' - f \frac{\partial f'}{\partial \underline{v}'} \right)$$

$$= \frac{-2\pi e^4}{m^2} \ln\Lambda \int d^3v \int d^3v' \, m\underline{v} \cdot \frac{\left(\underline{\underline{I}} - \hat{g}\hat{g}\right)}{g} \cdot \left(\frac{\partial f}{\partial \underline{v}} f' - f \frac{\partial f'}{\partial \underline{v}'} \right)$$

( ) antisymm

now symmetrizing:

$$\left.\frac{dE}{dt}\right|_{e,e} = -\frac{2\pi e^4}{m^2}\ln\Lambda \int d^3v \int d^3v' \; m\underbrace{(\underline{v}-\underline{v}')}_{\text{( ) antisymm}} \cdot \left(\frac{\underline{\underline{I}} - \vec{g}\vec{g}}{g}\right) \cdot \left(\frac{\partial f}{\partial \underline{v}} f' - f \frac{\partial f'}{\partial \underline{v}'}\right)$$

but $\underbrace{(\underline{v}-\underline{v}')}_{\sim \underline{g}} \cdot \underbrace{(\underline{\underline{I}} - \vec{g}\vec{g})}_{\text{transverse to } \underline{g}} = 0$

so

$$\left.\frac{dE}{dt}\right|_{e,e} = 0$$

show: ① $\left.\frac{dE}{dt}\right|_{e,i} \neq 0$

② relate to L.-B.

→ General Structure of Transport Problem

– by either $\begin{cases} \text{Boltzmann + small deflection} \\ \text{Fokker-Planck (Kulsrud)} \end{cases}$ (class)

⇒

arrive at:

$$\frac{\partial \langle f \rangle}{\partial t} = -\frac{\partial}{\partial \underline{v}} \cdot \underline{J}(\underline{v})$$

$$J(\underline{v}) = \sum_{spec} \int d^3v' \, B_{\alpha\beta} \left[\frac{\partial f(v')}{\partial v'_\beta} f(\underline{v}) - f(v') \frac{\partial f(\underline{v})}{\partial v_\beta} \right]$$

$$B_{\alpha\beta} = \frac{2\pi (qq')^2}{m^2 |v - v'|} \ln \Lambda \left[\delta_{\alpha\beta} - \frac{v_\alpha^{rel} v_\beta^{rel}}{v_{rel}^2} \right]$$

Now can re-group in form: (test distribution evolution)

$$\left. \frac{\partial \langle f_T \rangle}{\partial t} \right|_c = + \sum_{species, \alpha} \left\{ -\frac{\partial}{\partial v_j} \left(\frac{\partial h_\alpha}{\partial v_i} f_{Test} \right) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} \left(f_{Test} \frac{\partial^2 g_\alpha}{\partial v_i \partial v_j} \right) \right\}$$

$$* \left(\frac{4\pi n_\alpha q_T^2 q_\alpha^2}{m_T^2} \ln \Lambda \right)$$

here:

$\left.\begin{matrix} h_\alpha \\ g_\alpha \end{matrix}\right\}$ "Rosenbluth potentials" in terms $f(\underline{v}_\alpha)$

↳ distribution of field particles

where: $h_\alpha(\underline{v}) = \frac{m_T}{u_{\alpha}} \int d^3v' \, \frac{f_\alpha(\underline{v}')}{|\underline{v}-\underline{v}'|}$

$$g_\alpha(\underline{v}) = \int d^3v' \, f(\underline{v}') \, |\underline{v}-\underline{v}'|$$

Now can: - choose field distribution (i.e. background)
- set test problem (i.e. beam)

⇒ compute evolution - i.e. beam slow down.

Will consider several examples...

Applications of Collision Theory

Recall, have Landau - Fokker-Planck - Rosenbluth Equation:

$$\left.\frac{\partial \langle f \rangle}{\partial t}\right)_c = \sum_{\substack{\text{spcs} \\ \alpha}} -\frac{\partial}{\partial v_i}\left[\left\{ \frac{\partial h_\alpha}{\partial v_i} f_{test} - \frac{1}{2}\frac{\partial}{\partial v_j}\left(\frac{\partial^2 g_\alpha}{\partial v_i \partial v_j} f_{Test}\right)\right\} \cdot \left(\frac{4\pi \Lambda_\alpha}{m} q_T q_\alpha \ln \Lambda\right)\right]$$

$\alpha \rightarrow$ field particle species (i.e)
$T \rightarrow$ test particle species

$\left.\begin{array}{l} h_\alpha \\ g_\alpha \end{array}\right\}$ Rosenbluth potentials ≡ functionals of field particle distribution $f(\alpha)$

$$h_\alpha(\underline{v}) = \frac{m_T}{\mu_\alpha}\int d^3v' \frac{f_\alpha(\underline{v}')}{|\underline{v}-\underline{v}'|}$$

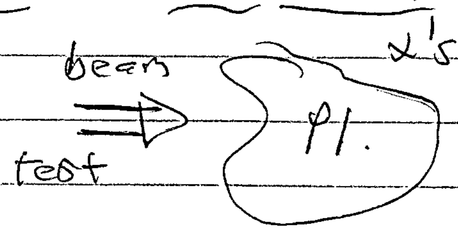
$$g_\alpha(\underline{v}) = \int d^3v' \, f(\underline{v}') \, |\underline{v}-\underline{v}'|$$

Now, consider applications:

(a) $\Rightarrow$ deceleration of beam
(b) $\Rightarrow$ spreading of a beam
(c) $\Rightarrow$ Conductivity
(d) $\Rightarrow$ Runaway (the exception)

(a) Deceleration of Beam

Consider



$$f_T(\underline{v}, t=0) = \delta(\underline{v} - \underline{v}_0)$$

Now, for deceleration, need: $\left.\dfrac{\partial}{\partial t} \underline{V}\right|_{\text{macro}}$

So taking $\int d^3v\ \underline{v}$ of F-P $\Rightarrow$

$$\boxed{\frac{\partial \underline{V}}{\partial t} = \sum_{\alpha} \gamma_{\alpha} \int d\underline{v}\ \overline{n}_{\alpha} \left(f_T \frac{\partial h_{\alpha}}{\partial \underline{v}} \right)}$$

$\Rightarrow$ sum of:
b-e drag
+
b-i drag

$$\gamma_{\alpha} = 4\pi Z_T^2 q_{\alpha}^2 \ln\Lambda / m_T^2$$

$$h_{\alpha}(\underline{v}) = \frac{m_T}{\mu_{\alpha T}} \int d^3v'\ \frac{f_{\alpha}(\underline{v}')}{|\underline{v} - \underline{v}'|}$$

Now,

$$\int d\underline{x}\, e^{-x^2}/|x-y| = \phi(y)\,\pi^{3/2}\, y$$

$$\phi(y) = \frac{2}{\sqrt{\pi}}\int_0^y e^{-x^2}\,dx$$

useful identity (Field Maxwellian)

⇒ plugging:

$$\frac{\partial V}{\partial t} = \left(\frac{4\pi n_0 e^2 Z_T^2 \ln\Lambda}{m_T^2}\right)\frac{\partial}{\partial V}\left[\underbrace{\frac{1}{V}\phi(u/v_{Te})\left(1+\frac{m_T}{m_e}\right)}_{b-e}\right.$$

$$\left. + \underbrace{\frac{Z}{V}\phi(u/v_{Ti})\left(1+\frac{m_T}{m_i}\right)}_{b-i}\right]$$

can define "slowing down time" by:

$$\tau_s = -V\left(\frac{\partial V}{\partial t}\right)^{-1}$$

so

→ if $V > v_{Te} > v_{Ti}$

→ b-e scattering dominates slowing down

so

$$\tilde{\tau}_s \approx M_T V^3 \Big/ 4\pi n_e e^2 q_T^2 \left(2+M_T/m_e\right) \ln\Lambda$$

- slowing & scattering longer for heavier
- insensitive temp

⇒ if $v_{Te} > V > v_{Ti}$

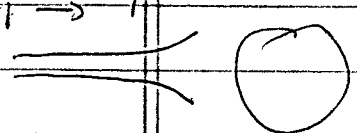
$$\tilde{\tau}_s = M_T^2 \Big/ 4\pi n_e e^2 q_T^2 \left[\frac{Z}{u^3}\left(1+\frac{M_T}{m_i}\right) + \frac{4}{\sqrt{3\pi}}\left(1+\frac{M_T}{m_e}\right)\frac{1}{v_{Te}^3}\right]\ln\Lambda$$

Temp dependence

$V < v_{Ti} < v_{Te} \Rightarrow$ HW.

(b) Spreading of Beam

$\parallel \rightarrow \quad \uparrow \perp$



$\partial_t V_\parallel^2$

$\partial_t V_\perp^2$

$$f_T = \delta(\underline{V}-\underline{v}) = \delta(\underline{V}_\parallel - \underline{v})$$

⇒ so, plugging into $\displaystyle \int d\underline{v}\; \underbrace{V^2}_{V_\parallel^2,\ V_\perp^2} \frac{\partial}{\partial t} f\Big)_c$

gives:

⇒ plugging in:

(∥ diffn)

$$\partial_t V_\parallel^2 = \frac{4\pi n_e e^3 q_T^2 \ln\Lambda}{m_T^2}\left[\frac{\partial}{\partial V_\parallel^2}\int d\underline{v}'(f_e + Zf_i)\,|\underline{V}-\underline{v}'|\right.$$

$$\left. + \; 2V_\parallel \frac{\partial}{\partial V_\parallel}\sum_\alpha\left(1+\frac{m_T}{m_\alpha}\right)\int d\underline{v}' \frac{Z_\alpha f_\alpha(\underline{v}')}{|\underline{V}-\underline{v}'|}\right]$$

and

$$\partial_t V_\perp^2 = \frac{4\pi n_e e^2 q_T^2 \ln\Lambda}{m_T^2}\left(\frac{\partial^2}{\partial V_{\perp_2}^2} + \frac{\partial^2}{\partial V_{\perp_3}^2}\right) *$$

$$\int d\underline{v}'\,(f_e + Zf_i)\,|\underline{V}-\underline{v}'|$$

note:

→ spread in $V_\parallel$ due drag + diffusion

$$\text{drag} \to \text{slow}, \quad \text{diffusion} \to \text{dispersion}$$

→ spread in $V_\perp$ due diffusion, only.

Now, can define deflection time by:

$$\tau_D = V^2 / dV_\perp^2/dt$$

can similarly consider:

→ T_D for $V \gg V_{Te}, V_{Ti}$ (supra-termal)

→ $V \sim V_{Ti} < V_{Te}$

→ $V \sim V_{Te} > V_{Ti}$

for supra-thermal:

$$T_D \sim m_T V^3 / 16\pi n_e e^2 Z_T^2 \ln\Lambda$$

others: HW

Ⓒ Conductivity / Resistivity due Collisions

Now, for conductivity

$$\frac{\partial f_e}{\partial t} - \frac{|e|}{m} E \frac{\partial f_e}{\partial V} = \left.\frac{\partial f_e}{\partial t}\right)_c$$

collision operator

stationarity ⇒

$$\frac{|e|}{m} E \int d^3V\, f_e = \int d^3V\, V \left.\frac{\partial f_e}{\partial t}\right)_c$$

electron acceleration balanced by collisional drag.

better to:

→ go to electron frame

→ consider ion motion

i.e

$$q_i E = -\frac{\partial}{\partial t}(m_i u_i)_c$$

take $f_i = \delta(\underline{v} - \underline{V})$

then

{ cold ion approx → use beam calcn

$$q_i E = -\frac{4\pi n e^2 q_i^2}{m_i} \ln\Lambda \frac{\partial}{\partial V}\left[\frac{1}{V}\phi(V/v_{Te})\left(1+\frac{m_i}{m_e}\right)\right]$$

→ point is that:

- ions are test
- no ion-ion scattering ($V > v_{Ti}$)
  just i-e scattering

80) for $V < V_{Te}$, can expand:

$$E = \frac{4\pi n_e e^2 q_i}{m_i}\left[\frac{1}{V_{Te}^3}\frac{4}{3\sqrt{\pi}}V\right]\frac{m_i}{m_e}\ln\Lambda$$

$$= \frac{n Z_i V}{\sigma} = \frac{J}{\sigma} \qquad \text{(frame)}$$

$$\boxed{\sigma = \frac{3 m_e}{16\sqrt{\pi}\, Z e^2 \ln\Lambda} V_{Te}^3}$$

$$\sigma \sim \frac{T_e^{3/2}}{Z e^2 \ln\Lambda\, m_e^{1/2}}$$

~ increases with T

~ indep. n

charge carriers ↑

but scattering ↑

$$\eta \sim 1/\sigma$$

$$\sim n/T_e^{3/2}$$

⇒ basic collisional conductivity/resistivity

!! goal !

(d) Runaways $\Rightarrow$ message is that stationary state not always possible for strong perturbation,

$\Rightarrow$ heuristics

- consider electron at thermal speed v in E

- in one mft, speed V is:

$$V \sim \frac{qE}{m_e} \tau_{mft} \sim \frac{qE}{m_e} \frac{l_{mfp}}{V_{Th}}$$

$$\sim \frac{qE}{m_e} \left(\frac{1}{N_e \sigma} \right) V_{Th}$$

but $\boxed{\sigma \sim \frac{4\pi e^4 \ln\Lambda}{m_e^2 V_{Th}^4}}$ (e-e)

$$V \sim \frac{qE}{m_e} \frac{1}{\left(N_e \frac{4\pi e^4 \ln\Lambda}{m_e^2 V_{Th}^4} V_{Th} \right)} \sim \frac{V_{Th}^3 m E}{4\pi e^3 \ln\Lambda N_e}$$

now, if $V_{Th} \sim V_{crit} \sim \left(\frac{4\pi e^3 \ln\Lambda N_e}{m E} \right)^{1/2}$

critical velocity

(really $E > E_{crit}$)

then $V \sim V_{Th}$

i.e. $V \sim (V_{Th}/V_c)^2 V_{Th}$

if $V_{Th} > V_c$, $\Rightarrow$ $\bar{V}$ replaces V_{Th} !

so:

momentum acquired is: (1 mft)

$$\Delta p \sim \frac{eE\ell}{V} \sim \frac{eE}{V N_e \sigma_e(V)} \sim \frac{V^3 m^2 E}{4\pi e^3 \ln\Lambda\, N_e}$$

$$\sim m\bar{V}\,(V/V_c)^2$$

$\Rightarrow$ $\Delta p \sim m\bar{V}$

$\Rightarrow$ electron accelerated w/o limit, if speed high enough !

$\Rightarrow$ if $V_c \lesssim V_{Th}$ $\Rightarrow$ bulk "runs away"

$\Rightarrow$ $\boxed{E > E_c = 4\pi e^3 \ln\Lambda\, N_e / T_e}$

$\Rightarrow$ Dreicer Field

$\Rightarrow$ if $V_c > V_{Th}$ $\Rightarrow$ only tail runs away

$E < E_c$
Dreicer

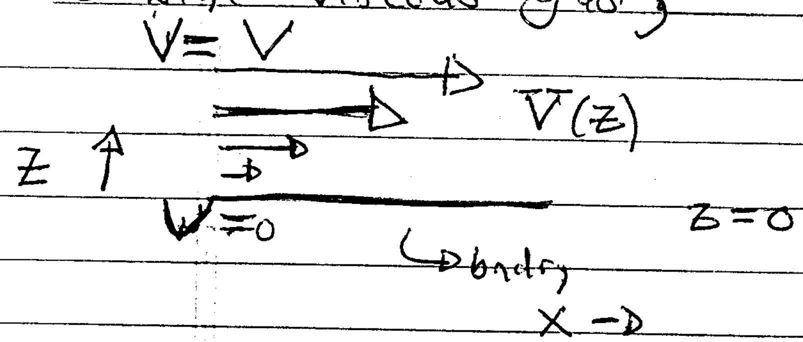
Now, if distribution parameters have gradients ∇T, ∇n, ∇V $\rightarrow$ relaxation and fluxes

$\Rightarrow$ How Compute with Boltzmann Eqn. ?

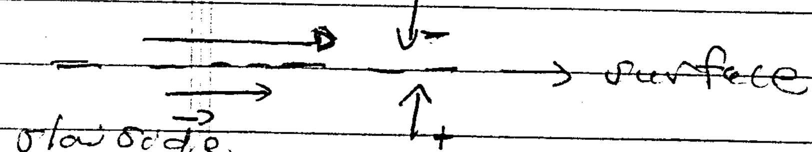
$\rightarrow$ Chapman - Enskog Expansion

- Simple physics of transport coefficients :

Consider viscous gas ;



$\rightarrow$ For transport of P_x thru imaginary surface,



$$\Gamma_+ = \int_+ dv\, m v_x v_z\, f \sim v_{Th} V n m_+$$

$$\Gamma_- = \int_- dv\, m v_x v_z\, f \sim v_{Th} V n m_-$$

define domains of integration yielding +, - flux

At first glance, would appear $\Gamma_+ = \Gamma_-$, but

→ minimum "scale of resolution" for imaginary surface is l_{mfp} ⇒ defines effective thickness

i.e. — — — — ⇒ ///////// l_{mfp} → $\bar{V}(z)\hat{x}$ has gradient across this

$$\Rightarrow \Gamma \cong - V_{Th}\, n m \left[\bar{V}\left(z + \frac{l_{mfp}}{2}\right) - \bar{V}\left(z - \frac{l_{mfp}}{2}\right) \right]$$

$$\cong - n m V_{Th}\, l_{mfp} \frac{\partial \bar{V}_x(z)}{\partial z}$$

i.e. $\cong - \eta \dfrac{\partial V_x(z)}{\partial z}$

$$\eta \cong n m \underbrace{V_{Th}\, l_{mfp}}_{D}$$ → shear viscosity

→ key points:

- equal # random walkers in +, − direction but
- more momentum (velocity gradient) brought in from above

⇒ viscous momentum transport.

→ Calculation

Consider Boltzmann equation for stationary flow

i.e.

$$\cancel{\frac{\partial f}{\partial t}} + \underset{\substack{\downarrow \\ V_{Th}/L_f}}{\underline{v} \cdot \underline{\nabla} f} = \underset{\substack{\downarrow \\ \nu_{eff} = \frac{V_{Th}}{l_{mfp}}}}{C(f)}$$

Seek :

$$\underset{\substack{\downarrow \\ \text{Z -direction flux of momentum in x direction}}}{\tilde{T}_{ZX}} = \int d^3v \underset{\substack{\downarrow \\ \text{flux in Z-direction}}}{v_z} \underbrace{v_x m f(\underline{x}, \underline{v})}_{\hat{x} \text{ momentum}}$$

How calculate ?

- Premise of Chapman - Enskog expansion.

$l_{mfp} \ll L \iff \nu_{eff} \gg \frac{V_{Th}}{L}$

⇒ - f remains very close to Maxwellian (i.e. equilibrium)

- seek calculate deviation from equilibrium i.e.

$$f = f_0 + \delta f$$

f_0 ~local Maxwellian; δf ↳ perturbation

$$\underline{v} \cdot \underline{\nabla} (f_0 + \delta f) = C(f_0 + \delta f)$$

l.o. $\Rightarrow$ $C(f_0) = 0$

$$\therefore \quad f_0 = f_{0\,max}$$

1st order $\quad \underline{v} \cdot \nabla f_{0\,max} = C(\delta f)$

$$\therefore \quad \delta f = C^{-1}\left[\underline{v} \cdot \nabla f_{0\,max}\right]$$

and thus

contribution vanishes by symmetry

$$\tau^{zx} = \int d^3v \; v_z \, v_x \, m \left[f_{0\,max} + \delta f \right]$$

$$= \int d^3v \; v_z \, v_x \, m \; C^{-1}\left[\underline{v} \cdot \underline{\nabla} f_{0\,max}\right]$$

$$\simeq -\eta \frac{\partial V_{0x}(z)}{\partial z}$$

↳ proportionality between $\left\{ \begin{array}{l} \text{flux} \\ \text{gradient} \end{array} \right.$

To calculate η:

$$C(\delta f) = -\nu(f - f_0)$$ "Crook" model (Krook)

$$= -r\,\delta f$$

$\hookrightarrow$ $r = r(v)$

$$\therefore\ \delta f = -\frac{1}{r}\,\underline{v}\cdot\nabla f_{0\,max}$$

$\rightarrow$ from $\bar{V}(z)\,\hat{x}$

$$= -\frac{1}{r}\,v_z\,\frac{\partial}{\partial z} f_0(\underline{v}, z)$$

$$\rightarrow\ \tilde{T}_{zx} = -\int d^3v\ v_z v_x \frac{m}{r} v_z \frac{\partial}{\partial z} f_0$$

$$f_0 = \frac{n(x)}{\left(\sqrt{2\pi}\,v_{Th}\right)^3}\exp\left[-\frac{m\left(\underline{v} - \bar{V}(z)\,\hat{x}\right)^2}{2T}\right]$$

{retain linear response only}

$$\therefore\ \tilde{T}_{zx} \cong -\#\, n m \frac{v_{Th}^2}{r}\frac{\partial}{\partial z}\bar{V}_x(z)$$

$$T_{zx} = -\#\, n m\, v_{Th}\, l_{mfp}\frac{\partial}{\partial z}\bar{V}_x(z)$$

$$= -\#\, n m\, D_{eff}\frac{\partial}{\partial z}\bar{V}_x(z)$$

Note:

$\Rightarrow$ $\underset{\substack{\downarrow \\ \text{transp.} \\ \text{coeff}}}{\eta} \sim nm \dfrac{l_{mfp}^2}{\tau} \sim nm \underset{\substack{\downarrow \\ \text{diffn} \\ \text{coeff.}}}{D}$ $\left\{ \begin{array}{l} 1/\tau = \nu_{eff} \Rightarrow \Delta T \sim \nu^{-1} \\ L \sim l_{mfp} \end{array} \right.$

$\Rightarrow$ $\underset{\substack{\downarrow \\ \text{themodynamic flux} \\ \text{(macroscopic)}}}{\tilde{T}^{zx}} = \underbrace{-\# nm\, D}_{\text{u-processes}} \dfrac{\partial V_x(z)}{\partial z}$

$\to$ gradient in f

$\hookrightarrow$ themodynamic force (deviation from max. entropy state) (macroscopic)

Constitutive relations $\leftrightarrow$ thermodynamic flux - force relation

$\Rightarrow$ In general, have vector relation

$$\underline{\Gamma} = -\underline{\underline{K}} \cdot \underline{\nabla} G$$

$\Rightarrow$ $\dfrac{\partial V_x(z)}{\partial z}$ drives thermal flux, too. etc

$\underline{\Gamma}$ $\to$ vector of fluxes

$\underline{\nabla} G$ $\hookrightarrow$ vector of gradients

$\underline{\underline{K}}$ $\to$ matrix of transport coefficients. (Onsager Matrix) $\to$ symmetric.

$\Rightarrow$ flux down gradient $\Leftrightarrow$ relaxation to f_{max} (H-thm.)

→ can have flux up a particular gradient, so long as total entropy production is > 0

Chemotaxis

(b.) Applications – Dynamic Screening – { Collective Enhancement of Collisional Relaxation

Consider form $B_{\alpha\beta}$:

$$B_{\alpha\beta} = 2(q q')^2 \int_{-\infty}^{+\infty} \int_{k<k_{max}} \delta(\omega - \underline{k}\cdot\underline{v})\,\delta(\omega-\underline{k}\cdot\underline{v}')\, \frac{k_\alpha k_\beta\, d^3k\, d\omega}{k^4 |\epsilon(\underline{k},\omega)|^2}$$

i.e. $\underline{k}\cdot\underline{v} = \underline{k}\cdot\underline{v}' \Rightarrow \underline{k}\cdot(\underline{v}-\underline{v}') = 0$

Consider stable, 2-species plasma. Then, have two collective resonances (i.e. weakly damped waves (no shift)):

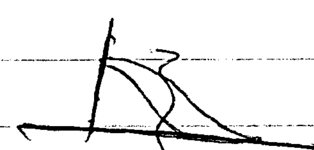
① → electron plasma waves; $\omega/k > V_{Te}$

② → ion acoustic waves; $V_{Ti} < \frac{\omega}{k} < V_{Te}$
(no shift fe)

① → tail of $\langle f \rangle_e$ ⇒ relatively few particles, little role in collision dynamics

② → if $T_e \gg T_i$

not



weak damping
$\langle f \rangle$ small

heavy ion, electron damping

∴ for $T_e \gg T_i$, ion acoustic resonance may enhance collisional relaxation (weakly damped modes)

To show, exploit 'pole approximation':

(collective resonance enhancement of $\underline{B}$)

$$\frac{1}{|\epsilon|^2} = \frac{1}{|\epsilon_r|^2 + |\epsilon_{IM}|^2}$$

$$\cong \frac{1}{\left[\left((\omega-\omega_k)\frac{\partial \epsilon}{\partial \omega}\Big|_{\omega_k}\right)^2 + |\epsilon_{IM}|^2\right]}$$

(damping → resonance linewidth)

$$\cong \frac{\pi}{|\epsilon_{IM}|}\,\delta(\epsilon_r)$$

↓ wave resonance

i.e. $\epsilon_r = 0$ → resonance location
ϵ_{IM} → resonance ~~size~~ width

and note for electron - electron collisions:

$\omega \ll \underline{k}\cdot\underline{v}$, $\underline{k}\cdot\underline{v}'$; due $\omega < \underline{k}\cdot \underline{v}_{Te}$ ordering

⇒

$$B_{\alpha\beta} \cong 2\pi q^4 \int_{-\infty}^{+\infty}\int \delta(\underline{k}\cdot\underline{v})\,\delta(\underline{k}\cdot\underline{v}')\,\delta(\epsilon_r)\,\frac{k_\alpha k_\beta}{|\epsilon_{IM}|}\,d^3k\,d\omega$$

Change variables:

$\kappa = \underline{k}\cdot\hat{n}$ (kappa) $\quad \hat{n}$ unit along $\underline{v}\times\underline{v}'$
$k_1 = \underline{k}\cdot\underline{v}$
$k_2 = \underline{k}\cdot\underline{v}'$

then: $\quad d^3k = d\kappa\, dk_1\, dk_2 / |\underline{v}\times\underline{v}'|$

so

$\Rightarrow$

$$B_{\alpha\beta} = \frac{2\pi e^4 n_\alpha n_\beta}{|\underline{v}\times\underline{v}'|}\, 2\int_{k>0} dk \int_{-\infty}^{+\infty} d\omega\, \frac{\delta(\epsilon_r(k,\omega))}{k^2\,|\epsilon_{IM}|}$$

i.e. collapse k_1, k_2 integrals

Now, $\epsilon_r = 1 - \dfrac{\omega_{pi}^2}{\omega^2} + \dfrac{1}{k^2\lambda_D^2}$

$\omega = kc_s \big/ \left(1+k^2\lambda_{De}^2\right)^{1/2}$

$\Bigg\}$ ion-acoustic wave

$$\Rightarrow \quad \epsilon_{IM} = \sqrt{\frac{\pi}{2}}\,\frac{\omega}{k^3}\Bigg(\underbrace{\frac{1}{\lambda_{De}^2 V_{Te}}}_{\substack{\text{electron L.D.}\\ \omega\ll kV_{Te}}} + \underbrace{\frac{1}{\lambda_{Di}^2 V_{Ti}}\, e^{-\omega^2/2k^2V_{Ti}^2}}_{\text{I.L.D}}\Bigg)$$

dominant contribution from shortwavelength ($k\lambda_{De}>1$)

$\rightarrow$ (i.e. $\underline{\underline{\text{max}}}$ $1/|\epsilon_{IM}|$)

$$\therefore \begin{cases} \omega \cong \omega_{pi} \\ \epsilon_{IM} = \sqrt{\dfrac{\pi}{2}}\,\dfrac{\omega_{pi}}{k^3}\left(\dfrac{1}{\lambda_{De}^2 V_{Te}} + \dfrac{1}{\lambda_{Di}^2 V_{Ti}}\, e^{-1/k^2\lambda_{Di}^2}\right) \end{cases}$$

$\Rightarrow$

$$\delta(\epsilon_r) = \delta\left(1-\omega_{pi}^2/\omega^2\right) \cong \frac{1}{2}\,\omega_{pi}\left[\delta(\omega-\omega_{pi}) + \delta(\omega+\omega_{pi})\right]$$

P $$\boxed{B_{\alpha\beta} = \frac{4\pi e^4 \omega_{pi}}{|\underline{v}\times\underline{v}'|} n_\alpha n_\beta \int \frac{dk}{k^2 |\epsilon_{I,I}(\omega_{pi}, k)|}}$$

$$\varepsilon = k^2 \lambda_{Di}^2$$

$$\epsilon_{I\Pi} = \sqrt{\frac{\pi}{2}}\, \frac{\omega}{k^3} \left\{ \frac{1}{\lambda_{De}^2 v_{Te}} + \frac{1}{\lambda_{Di}^2 v_{Ti}} e^{-\omega_{pi}^2/2k^2 v_{Ti}^2} \right\}$$

$$\therefore \left\{ \begin{aligned} B_{\alpha\beta} &= n_\alpha n_\beta \frac{2\sqrt{2\pi}\, e^4 v_{Te} \lambda_{De}^2}{|\underline{v}\times\underline{v}'|\, \lambda_{Di}^2} \int d\varepsilon \Big/ \left[1 + \exp\left(-\frac{1}{2\varepsilon} + \frac{L_1}{2} \right) \right] \\ L_1 &= \ln\left((T_e/T_i)(v_{Te}^2/v_{Ti}^2) \right) \end{aligned} \right.$$

Now:

(i) $v_{Ti} < \frac{\omega}{k} < v_{Te} \Rightarrow \cancel{1 <} \varepsilon < 1$ $\quad (\omega_{pi}\cancel{\lambda_{Di}}/\omega_{pe}\lambda_{De})^2$

(ii) $L_1 \gg 1 \Rightarrow$ expand $O(1/L_1)$

i.e. dominant contribution when:

$$\exp\left(-1/2\varepsilon + \frac{L_1}{2}\right) \lll 1 \Rightarrow \varepsilon \leq 1/L_1$$

note: $\left[1 + \exp\left(-1/2\varepsilon + L_1/2\right)\right]^{-1} \equiv$ i.e. denominator yields;

$$\boxed{B_{\alpha\beta} = n_\alpha n_\beta \left[\frac{2\sqrt{2\pi}\, e^4 v_{Te} \lambda_{De}^2}{|v \times v'| \, \lambda_{Di}^2}\right](1/L)}$$

Now;

$$\underset{\substack{\text{(collective resonance}\\ \text{dominant)}}}{\overset{\substack{\text{above } (v \ll v_{Te})\\ \downarrow}}{B_{\alpha\beta} = B_{\alpha\beta}^{\text{collective}}}} + \underset{\substack{\text{(ballistic spectrum}\\ \text{dominant)}}}{\overset{(v \sim v_{Te})}{B_{\alpha\beta}^{\text{coulomb}}}}$$

as peaks in spectrum not coincident.

$$B_{\alpha\beta}^{\text{coulomb}} = \frac{2\pi e^4}{|v - v'|} L \left[\delta_{\alpha\beta} - \frac{(v_\alpha - v_\alpha')(v_\beta - v_\beta')}{|v - v'|^2}\right]$$

$$\cong \frac{2\pi e^4}{v_{Te}} L_{\text{coulomb}}$$

$$B^{\text{collective}} \geq B^{\text{coulomb}} \quad \text{if:}$$

$$\boxed{\frac{T_e}{T_i L_1} \geq \underbrace{L_{\text{coulomb}}}_{\text{Coulomb log.}}}$$

Criterion for dominance of collective effect enhanced scattering (need $T_e \gg T_i$)